EXTENSIONS OF THE HAUSDORFF-YOUNG THEOREM(1)

BY
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Dedicated to the memory of my nephew, K. Ramakrishna, who appeared to be so brilliant.

ABSTRACT
In this paper the clasical Hausdorff-Young theorem, which states that if $f \in L^p$, $1 \leqq p \leqq 2$, on the line and $\hat{f}$ is its Fourier transform, then $\|\hat{f}\|_q \leqq \|f\|_p$ where $q^{-1} + p^{-1} = 1$, is extended in two ways for certain Orlicz spaces L^Φ. If L^Φ is based on (G, μ), (1) an arbitrary compact topological group with Haar measure, and (2) a locally compact abelian topological group and μ is again the Haar measure, then the above inequality is extended to these cases. Various other related results and remarks are also included.

1. **Introduction.** If $L^p(R, \mu)$, $p \geqq 1$, is the Lebesgue space on the line R with μ as the Lebesgue measure, and if $f \in L^p(R,\mu)$ and $\hat{f} = Tf$, the Fourier transform of f, then the Hausdorff-Young theorem states that for all $1 \leqq p \leqq 2$, $\hat{f}$ exists and moreover $\|\hat{f}\|_q \leqq \|f\|_p$, where $p^{-1} + q^{-1} = 1$ and $\|\cdot\|_p$ is the norm in $L^p(R,\mu)$. This result is quickly proved from the Riesz convexity theorem using the facts that $\|\hat{f}\|_\infty \leqq \|f\|_1$ and (the Plancherel formula) $\|\hat{f}\|_2 = \|f\|_2$. It is known that Tf is not defined for all f in $L^p(R,\mu)$ if $p > 2$.

The purpose of this paper is to generalize this result with an enlargement of spaces for which the inequalities of Hausdorff-Young type hold. This is achieved by replacing the L^p-spaces by the L^Φ-spaces of Orlicz. In fact, simple examples can be constructed (cf., e.g., [5], p. 29) to show that there exist Young's functions Φ (hence L^Φ-spaces) that grow faster than $|x|$ but not x^2 and yet do not satisfy the so-called Δ_2-condition. A simple modification of this construction shows the extensiveness of the L^Φ-spaces 'between' any two L^p-spaces. This example of [5] is one of the motivations for the following work. The results obtained below include such spaces when the measure space (R,μ) is replaced by (G,μ) where G is either an arbitrary compact (Hausdorff) group or a locally compact abelian group, and μ is the (normalized) Haar measure on G. These results are obtained first by extending a key inequality of Hausdorff-Young, in the form of Hardy and Littlewood ([2], p. 170), to the present situation and then proving the general

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statements (Sections 3 and 4). The present treatment illuminates the nature of these inequalities. Also their dependence on the Plancherel theorem will be clarified. Another extension of the Hausdorff-Young theorem was given in [12] and its relation with the present work is discussed in the last section.

2. **Preliminaries.** In this section some notation and preliminaries will be set forth. They will be used freely later on. Let Φ and Ψ be non-trivial symmetric convex functions on the line, vanishing at the origin, and satisfying the Young's inequality and the normalization as follows:

$$xy \leqq \Phi(x) + \Psi(y), \quad \Phi(1) + \Psi(1) = 1. \tag{2.1}$$

Such functions are said to be a (normalized) complementary Young's pair. Let $L^\Phi(G,\mu)$ be the (sub-) space of (equivalence classes of) measurable scalar functions f on a measure space (G,μ) for which $J_\Phi(f) < \infty$, where

$$J_\Phi(f) = \inf\left\{k > 0: \int_G \Phi\left(\frac{|f|}{k}\right) d\mu \leqq \Phi(1)\right\}. \tag{2.2}$$

It is known that (2.2) is a norm and $L^\Phi(G,\mu)$, or L^Φ for short, is a Banach (or B-) space under this norm (cf. [15], Vol. I, p. 174 ff, or [5] in which the [inessential but convenient] normalization equality of (2.1) was not assumed). The space L^Φ is called an Orlicz space and Φ a Young's function. Hereafter, in this paper, G *will be a topological group* (either compact or locally compact abelian) *and μ the Haar measure on G*. Let $\mathcal{M}^\Phi$ denote the closed subspace of L^Φ determined by all bounded functions. Similar definitions hold for L^Ψ, $\mathcal{M}^\Psi$ where Ψ is complementary to Φ. Also Φ is called a continuous Young's function if it is continuous on the line and $\Phi(x) > 0$ whenever $x > 0$, (cf. [5]). For the usefulness of the normalization of (2.1), see [11].

A partial ordering between the Young's functions Φ_1, Φ_2 is needed below. It is defined as: $\Phi_1 \leqq \Phi_2$ whenever $\Phi_1(ax) \leqq b\Phi_2(x)$ for $|x| \geqq x_0$ and $\Phi_2(cx) \leqq d\Phi_1(x)$ for $|x| \leqq x_1$ where x_0, x_1, a, b, c and d are some fixed positive numbers independent of x. [In the case of L^p-spaces, one notes that, for all $p \geqq 1$, $a = b = c = d = 1$, $x_1 \geqq 1$ and $x_0 \geqq 1$.] This ordering implies the following inclusion relations between the Orlicz spaces (cf. [8], Theorems 4 and 5 on pp. 51–52): Let $\Phi_1 \leqq \Phi_2$ be continuous. Then the finiteness of the measure μ implies $L^{\Phi_1} \supset L^{\Phi_2}$ and $J_{\Phi_1}(\cdot) \leqq \alpha J_{\Phi_2}(\cdot)$ and if $\mu(\{x\}) = \delta \geqq \delta_0 > 0$ for each point x (so L^Φ becomes the sequence space l^Φ), one has $l^{\Phi_1} \subset l^{\Phi_2}$ and the corresponding norms satisfy $\|\cdot\|_{l^{\Phi_2}} \leqq \beta \|\cdot\|_{l^{\Phi_1}}$. [As shown in [8], the converse implications also hold for this ordering.]

LEMMA 2.1. *Let (Φ_i, Ψ_i), $i = 1,2$ be two complementary Young's pairs and $\Phi_1 \leqq \Phi_2$ be continuous. Then the ordering $\Psi_2 \leqq \Psi_1$ holds.*

Proof. First consider the relations near the origin. By definition $\Phi_1(ax) \leqq b\Phi_2(x), 0 < x \leqq x_0$. Let $\Phi(x) = 1/b\ \Phi_1(ax) \leqq \Phi_2(x)$. Then $\Psi_2(x) \leqq \Psi(x)$ for $0 < x \leqq y_0$ and for some $y_0 < \infty$ where Ψ is complementary to Φ. This is seen as follows. Using the integral representation of convex functions (cf. [5], p. 5),

$$\Phi(x) = \int_0^x p(t)\,dt \leqq \int_0^x p_2(t)\,dt = \Phi_2(x), \qquad 0 < x \leqq x_0, \tag{2.3}$$

where the p's are (unique) increasing functions. If q and q_2 are the inverses of p and p_2 of (2.3) then the point $(x, q_2(x))$ gives equality in Young's inequality (2.1) for Φ_2, Ψ_2 (and only inequality for Φ, Ψ). Hence

$$\Phi_2(q_2(x)) + \Psi_2(x) = |x|\,q_2(x) \leqq \Phi(q_2(x)) + \Psi(x),\ 0 < x \leqq y_0 = q_2(x_0). \tag{2.4}$$

(2.3) and (2.4) then imply $\Psi_2(x) \leqq \Psi(x)$ for $0 < x \leqq y_0 = q_2(x_0)$ at once. But $\Psi(x) = 1/b\,\Psi_1(bx/a)$ which is a consequence of ([5], p. 12). Hence, by a change of scale, it follows that $\Psi_2(ax/b) \leqq 1/b\,\Psi_1(x)$ for $0 < x \leqq ay_0/b$, proving the first inequality needed for $\Psi_2 \leqq \Psi_1$. The second inequality is immediate from Theorem 3.1 of ([5], p. 16). This proves the lemma.

3. Compact groups. A key lemma. In this section an extension of the Hausdorff-Young inequality will be established for (arbitrary) compact groups G, and for a class of Young's functions $\Phi \leqq \Phi_0$ where $\Phi_0(x) = \frac{1}{2}x^2$. These results are essential for all later extensions to the locally compact case. The key step in this work is a generalization of the proof of Hardy and Littlewood ([2], p. 170) which was given for the circle group and for the L^p, $1 < p < 2$, spaces. However, the result in the present case is considerably more difficult even though in both cases the basic idea is to use equality conditions in Hölder's inequalities appropriately.

Thus let G be a compact (Hausdorff) group. Let $\{U(\alpha)\}_{\alpha \in A}$ be (by the Peter-Weyl theorem) a complete set of unitary, inequivalent, irreducible (hence finite-dimensional) representations of G, where A is an index set. If $U_x(\alpha) = \{u_x(i,j,\alpha)\}_{i,j=1}^{d(\alpha)}$ is a matrix representation, relative to some basis where $d(\alpha) \geqq 1$ is the dimension of the representation, then for an $f \in L^\Phi(G,\mu)$ let $\hat{f}(i,j,\alpha) = \int_G f(x) u_x(i,j,\alpha)\,d\mu(x)$ where μ is the normalized Haar measure on G. $\hat{f}$ is called the Fourier transform of f and $\{u_x\}$ are orthogonal: i.e.,

$$d(\alpha) \int_G u_x(i,j,\alpha)\, u_x(i',j',\alpha')^*\, d\mu(x) = 1, \qquad \text{if } i = i',\ j = j',\ \alpha = \alpha',$$

and $= 0$ otherwise. Here 'star' denotes complex conjugation. Let

$$F^2(\alpha, f) = \sum_{i,j=1}^{d(\alpha)} |\hat{f}(i,j,\alpha)|^2 / d(\alpha), \quad \alpha \in A, \tag{3.1}$$

$$J_\Phi(F) = \inf\left\{k > 0: \sum_{\alpha \in A} \Phi\left(\frac{F(\alpha,f)}{k}\right) d^2(\alpha) \leqq \Phi(1)\right\}. \tag{3.2}$$

Then $J_\Phi(\cdot)$ is a norm for l^Φ, and if Φ is continuous then there is a $k_0 = J_\Phi(F)$ in (3.2) such that the inequality is an equality with $k = k_0$ on the left.

THEOREM 3.1. *Let (Φ,Ψ) be a continuous Young's complementary pair such that* (i) *$\Phi \leqq \Phi_0$ where $\Phi_0(x) = \frac{1}{2}x^2$,* (ii) *the derivative Ψ' of Ψ exists and satisfies $\Psi'(x) \leqq a_0 x^r$ for $0 \leqq x < \infty$ and some $r \geqq 1$. If G is a compact group and $f \in \mathcal{M}^\Phi(G,\mu)$, then one has*

$$J_\Psi(F) \leqq k_0 J_\Phi(f), \tag{3.3}$$

$$J_\Psi(f) \leqq k_1 J_\Phi(F), \tag{3.4}$$

where k_0 and k_1 are some constants depending only on the ordering constants of '$\leqq$' and Φ and a_0, and where the notations of (3.1) *and* (3.2) *are used in* (3.3) *and* (3.4).

REMARK. The right (and left) derivative of a convex function exists everywhere and they may be different at most at only a countable set of points. Since Ψ is continuous one may assume Ψ' to be continuous everywhere by a redefinition (e.g. joining the discontinuities by straight line segments; cf. [15], Vol. I, p. 25) and thus the assumption of the first part of (ii) above is made for convenience. A more general condition on (Φ,Ψ) will be given later in Theorem 3.3 of this section.

The proof of this theorem depends on the following *key lemma*:

LEMMA 3.2. *Let (Φ,Ψ) satisfy the conditions of Theorem* 3.1. *If $A_0 \subset A$ is a finite set define a function $f_{A_0}(\cdot)$ by*

$$f_{A_0}(x) = \sum_{\alpha \in A_0} d(\alpha) \sum_{i,j=1}^{d(\alpha)} c(i,j,\alpha) u_x(i,j,\alpha), \quad x \in G, \tag{3.5}$$

where $\{c(i,j,\alpha)\}$ are some complex constants. Then the inequalities (3.3) *and* (3.4) *hold for f_{A_0}. More explicitly:*

$$J_\Psi(F_{A_0}) \leqq \bar{k}_0 J_\Phi(f_{A_0}), \quad J_\Psi(f_{A_0}) \leqq \bar{k}_1 J_\Phi(F_{A_0}) \tag{3.6}$$

where for $\alpha \in A_0$, $F^2_{A_0}(\alpha) = \sum_{i,j=1}^{d(\alpha)} |c(i,j,\alpha)|^2/d(\alpha)$, and $\bar{k}_i$, $i = 0,1$, are some constants depending only on Φ and the ordering $\Phi \leqq \Phi_0$.

Proof. Since the computations are somewhat involved, the details will be presented here in steps.

I. Let $A_0 \subset A$ and f_{A_0} be as in the lemma. It is clear that $\hat{f}_{A_0}(i,j,\alpha) = c(i,j,\alpha)$, if $\alpha \in A_0$, and $= 0$ if $\alpha \notin A_0$. So $F_{A_0}(\alpha)$ of (3.6) is simply $F(\alpha, f_{A_0})$ of (3.1) which vanishes for $\alpha \notin A_0$, and to avoid confusion, set $J_\Phi(F(\cdot,f_{A_0})) = S_\Phi(f_{A_0})$. In what follows A_0, and Φ (hence Ψ) are fixed. If $0 \neq f \in L^\Phi$ and $\tilde{c}(i,j,\alpha)$ are the Fourier coefficients of f, let $\tilde{f}_{A_0}$ be the corresponding function in (3.5) with these c's. Define (as in [2]),

$$M = M_\Phi(A_0) = \sup\{S_\Psi(\tilde{f}_{A_0})/J_\Phi(f) : f \neq 0\}, \tag{3.7}$$

$$M' = M'_\Phi(A_0) = \sup\{J_\Psi(f_{A_0})/S_\Phi(f_{A_0}) : f_{A_0} \text{ of (3.5) with all } c\text{'s}\}. \tag{3.8}$$

It will be proved that M and M' are finite and equal. Since the quantities here are ratios of norms, it may and will be assumed $S_\Psi(\tilde{f}_{A_0}) = 1$, for convenience. This entails, from the continuity of Ψ and (3.2) (with $k_0 = 1$ there),

$$\sum_{\alpha \in A} \Psi(F(\alpha, \tilde{f}_{A_0}))\, d^2(\alpha) = \Psi(1).$$

The left side is zero if $\alpha \in A - A_0$. Since $0 < \Psi(1) < 1$, at least one of the terms on the left exceeds $[\Psi(1)/\bar{a}]$, where $\bar{a}$ is the (finite) number of elements of A_0. If that term corresponds to $\alpha_0 \in A_0$, then

$$0 < \Psi^{-1}\left[\frac{\Psi(1)}{\bar{a}\, d^2(\alpha_0)}\right] \leqq F(\alpha_0, \tilde{f}_{A_0}). \tag{3.9}$$

Moreover, from definition in (3.1) one has

$$\begin{aligned} F(\alpha, \tilde{f}_{A_0}) &\leqq [d(\alpha)]^{-1/2} \sum_{i,j=1}^{d(\alpha)} |\tilde{c}(i,j,\alpha)| \\ &\leqq [d(\alpha)]^{-1/2} \int_G |f(x)| \sum_{i,j=1}^{d(\alpha)} |u_x(i,j,\alpha)|\, d\mu(x) \\ &\leqq [d(\alpha)]^{-1/2} \int_G |f(x)| \left[\sum_{i,j=1}^{d(\alpha)} |u_x(i,j,\alpha)|^2\right]^{1/2} d(\alpha)\, d\mu(x) \\ &= d(\alpha) \int_G |f(x)|\, d\mu(x) \leqq d(\alpha) J_\Phi(f). \end{aligned} \tag{3.10}$$

Here once the Schwarz's inequality and then the Hölder inequality (with $J_\Psi(1) = 1$) are used. (3.9) and (3.10), with $\alpha = \alpha_0$, yield

$$\frac{1}{J_\Phi(f)} \leqq \left[d(\alpha_0)/\Psi^{-1}\left(\frac{\Psi(1)}{\bar{a} d^2(\alpha_0)}\right)\right] < \infty. \tag{3.11}$$

Since the right side of (3.11) is independent of f, taking the supremum on the left gives M so that $M < \infty$ holds.

II. To show that M' is also finite, actually $M' \leqq M$ will be proved. For this purpose define the function g, from f_{A_0} of (3.5) as,

$$g(x) = \Psi'\left(\frac{|f_{A_0}(x)|}{J_\Psi(f_{A_0})}\right) \operatorname{sgn}(f_{A_0}(x)). \tag{3.12}$$

It follows from ([15], p. 175) that $J_\Phi(g) = 1$, and that there is equality in the Hölder inequality between f_{A_0} and g. So from this, the Parseval formula, and (3.7) one has

$$J_\Psi(f_{A_0}) = \int_G g(x) f_{A_0}^*(x)\, d\mu(x) = \sum_{\alpha \in A_0} d(\alpha) \sum_{i,j=1}^{d(\alpha)} \hat{g}(i,j,\alpha) \hat{f}_{A_0}(i,j,\alpha)^*$$

$$(3.13) \qquad \leqq \sum_{\alpha \in A_0} d^2(\alpha) F(\alpha, \tilde{g}_{A_0}) F(\alpha, f_{A_0}) \leqq S_\Phi(f_{A_0}) S_\Psi(\tilde{g}_{A_0})$$

$$\leqq S_\Phi(f_{A_0}) M J_\Phi(g) = M S_\Phi(f_{A_0}).$$

Dividing throughout by $S_\Phi(f_{A_0})$ and taking supremum over all c's in f_{A_0}, it follows that $M' \leqq M < \infty$.

III. It will next be shown that $M \leqq M'$ so that $M = M'$ will follow. For this purpose define a function h, from $\tilde{f}_{A_0}$ for an $0 \neq f \in L^\Phi$, considered in (3.7), as follows:

$$(3.14) \qquad h_{A_0}(x) = \sum_{\alpha \in A_0} d(\alpha) \tilde{\Psi}(F(\alpha, \tilde{f}_{A_0})) \tilde{c}(i,j,\alpha)^* u_x(i,j,\alpha),$$

where $\tilde{\Psi}(x) = \frac{1}{x} \Psi'\left(\frac{x}{k}\right)$, for $x > 0$ with $\tilde{\Psi}(0) = 0$, and where $k = J_\Psi(F(\cdot, \tilde{f}_{A_0})) > 0$. Then the Parseval formula once again yields

$$\int_G f(x) h_{A_0}^*(x) h_{A_0}(x)\, d\mu(x) = \int_G \tilde{f}_{A_0}(x) h_{A_0}^*(x)\, d\mu(x)$$

$$= \sum_{\alpha \in A_0} d(\alpha) \sum_{i,j=1}^{d(\alpha)} \tilde{c}(i,j,\alpha) \cdot \tilde{c}(i,j,\alpha)^* \cdot \tilde{\Psi}(F(\alpha, \tilde{f}_{A_0}))$$

$$= \sum_{\alpha \in A_0} d^2(\alpha) F^2(\alpha, \tilde{f}_{A_0}) \cdot \tilde{\Psi}(F(\alpha, \tilde{f}_{A_0}))$$

$$= \sum_{\alpha \in A_0} d^2(\alpha) F(\alpha, \tilde{f}_{A_0}) \cdot \Psi'\left(\frac{F(\alpha, \tilde{f}_{A_0})}{k}\right) = S_\Psi(\tilde{f}_{A_0})$$

where, in the last line, the equality for Hölder's inequality and the definition $J_\Psi(F(\cdot, \tilde{f}_{A_0})) = S_\Psi(\tilde{f}_{A_0})$ are used. Thus

$$(3.15) \qquad S_\Psi(\tilde{f}_{A_0}) = \int_G f(x) h_{A_0}^*(x)\, d\mu(x) \leqq J_\Phi(f) \cdot J_\Psi(h_{A_0}) \leqq J_\Phi(f) M' S_\Phi(h_{A_0}).$$

However, a simple computation shows that $F(\alpha, h_{A_0}) = \Psi'\left(\frac{F(\alpha, \tilde{f}_{A_0})}{k}\right)$. From this it follows, as in (3.12), that $S_\Phi(h_{A_0}) = J_\Phi(F(\cdot, h_{A_0})) = 1$, so that (3.15) yields $[S_\Psi(\tilde{f}_{A_0})/J_\Phi(f)] \leqq M'$. Taking supremum over all f then gives, by (3.7), that $M \leqq M'$. This proves $M = M'$.

IV. It remains to show that M is bounded above. Let f_{A_0} be a maximal function in (3.7). Since the norms (and ratios) are continuous functions of the c's, such functions exist. Thus $M = M' = [J_\Psi(f_{A_0})/S_\Phi(f_{A_0})]$. This implies there is equality throughout in (3.13) yielding the following equation for the corresponding g, of (3.12),

$$S_\Psi(\tilde{g}_{A_0}) = M, \quad \text{and} \quad J_\Psi(f_{A_0}) = MS_\Phi(f_{A_0}). \tag{3.16}$$

Fix this g and f_{A_0} in what follows. Note also that M is positive. In fact taking the constant function $f = 1$, one notes that, by ([7], p. 159), it is a character of the compact group G. Since $J_\Phi(1) = 1$ and a simple computation shows that $S_\Psi(\tilde{f}_A) = \left[\Phi^{-1}\left(\frac{\Phi(1)}{d^2(\alpha_0)}\right)\right]^{-1} = k_0 \geqq 1$, for some $\alpha_0 \in A$ so that $M \geqq S_\Psi(\tilde{f}_{A_0})/J_\Phi(f) \geqq k_0 \geqq 1$.

For the upper bound of M, consider the Bessel inequality:

$$S_2^2(\tilde{g}_{A_0}) = \sum_{\alpha \in A_0} d^2(\alpha) \sum_{i,j=1}^{d(\alpha)} |\hat{g}(i,j,\alpha)|^2 \leqq \int_G |g(x)|^2 d\mu(x) \leqq J_\Psi(g^2). \tag{3.17}$$

(Here $J_\Psi(1) = 1$ is used in the last Hölder's inequality.) But letting $\Psi_1(x) = \Psi(x^2)$ which is a Young's function satisfying $\Psi \leqq \Psi_1$, and since g is a bounded function, one has, if $a^2 = J_\Psi(g^2)$,

$$\Psi_1(1) = \Psi(1) = \int_G \Psi\left(\frac{|g|^2}{a^2}\right) d\mu = \int_G \Psi\left(\frac{|g|}{a}\right) d\mu. \tag{3.18}$$

Thus $J_{\Psi_1}(g) = a$. This and (3.17) imply

$$S_2(\tilde{g}_{A_0}) \leqq J_{\Psi_1}(g). \tag{3.19}$$

Thus far however the hypothesis (ii) on the Young's function Φ (i.e., that $\Psi'(x) \leqq a_0 x^r$, $r \geqq 1$) has not been used. This will be utilized now to obtain the bound. The whole point here is to find a connection between the norm of g and that of f_{A_0}.

If $a = J_{\Psi_1}(g)$, (cf. (3.18)) then from the definition of norm it follows that there exists a $\beta_0 > 0$ such that

$$\begin{aligned} 1 &= \int_G \Psi_1\left(\frac{g}{a\beta_0}\right) d\mu = \int_G \Psi_1\left[\frac{1}{a\beta_0}\Psi'\left(\frac{|f_{A_0}|}{J_\Psi(f_{A_0})}\right)\right] d\mu \\ &\leqq \int_G \Psi_1\left[\frac{a_0}{a\beta_0}\left(\frac{|f_{A_0}|}{J_\Psi(f_{A_0})}\right)^r\right] d\mu, \text{ by hypothesis on } \Psi, \\ &= \int_G \Psi_2\left(\beta_1 \frac{|f_{A_0}|}{J_\Psi(f_{A_0})}\right) d\mu, \end{aligned} \tag{3.20}$$

where $\beta_1^r = \frac{a_0}{a\beta_0} > 0$, and $\Psi_2(x) = \Psi_1(x^r)$ which is a Young's function such that $\Psi \leqq \Psi_1 \leqq \Psi_2$. Then (3.20) implies the following crucial inequality. There exists a constant $\beta_2 > 0$, depending only on Ψ_2 and independent of f_{A_0}, such that

$$J_{\Psi_2}\left(\frac{\beta_1 f_{A_0}}{J_\Psi(f_{A_0})}\right) \geqq \beta_2 > 0.$$

Since $J_{\Psi_2}(\cdot)$ is a norm, one has, on recalling the definition of β_1 above,

$$\left[\frac{J_{\Psi_2}(f_{A_0})}{J_\Psi(f_{A_0})}\right]^r \geqq \beta_2^r\left(\frac{\beta_0}{a_0}\right) J_{\Psi_1}(g) = \beta_3 J_{\Psi_1}(g), \quad \text{(say)}. \tag{3.21}$$

Now collecting various estimates from (3.16), (3.19), (3.21) and using the fact that $\Psi_2 \geqq \Psi_1 \geqq \Psi \geqq \Psi_0$ (so $\Phi_0 \geqq \Phi \geqq \Phi_1 \geqq \Phi_2$ and $l^{\Phi_2} \subset l^{\Phi_1} \subset l^{\Phi} \subset l^2 \subset l^{\Psi} \subset l^{\Psi_1} \subset l^{\Psi_2}$ by Lemma 2.1 and the statement preceding it) one obtains the following important chain of inequalities:

$$\begin{aligned} 1 \leqq M = M_\Phi = S_\Psi(\tilde{g}_{A_0}) \leqq k_2 S_2(\tilde{g}_{A_0}) &\leqq \frac{k_2}{\beta_3}\left[\frac{J_{\Psi_2}(f_{A_0})}{J_\Psi(f_{A_0})}\right]^r \\ \leqq \frac{k_2}{\beta_3}\left[\frac{J_{\Psi_3}(f_{A_0})}{J_\Psi(f_{A_0})}\right]^r \leqq \frac{k_2}{\beta_3}\left[\frac{M_{\Phi_3}S_{\Phi_3}(f_{A_0})}{M_\Phi S_\Phi(f_{A_0})}\right]^r &\leqq \frac{k_2}{\beta_3}\left[\beta_4 \frac{M_{\Phi_3}}{M_\Phi}\right]^r \end{aligned} \tag{3.22}$$

where $S_{\Phi_3}(f_{A_0}) \leqq \beta_4 S_\Phi(f_{A_0})$ for some $\beta_4 > 0$ depending only on Φ_3 and Φ was used, and where $\Phi_3 \leqq \Phi_2$ is complementary to $\Psi_3, \Psi_3(x) = \frac{a_0}{r+1}|x|^{2r(r+1)} \geqq \Psi_2(x)$, for $x \geqq 0$. Thus (3.22) can be written as

$$1 \leqq M_\Phi^{r+1} \leqq k_3^r M_{\Phi_3}^r \tag{3.23}$$

for some $k_3 > 0$, which depends only on Φ, Φ_1, Φ_3 and the ordering constants. But $L^{\Psi_3} = L^q$ where $q = 2r(r+1) \geqq 2$, so that by ([2], p. 170, when G is the circle group and [4], for the general compact group G—(a short independent proof of the latter is given below as Proposition 3.4) it follows that $M_{\Phi_3} \leqq k_4 < \infty$, k_4 being a constant depending only on a_0 and r. This and (4.23) yield immediately

$$1 \leqq M_\Phi \leqq k_5 < \infty, \tag{3.24}$$

where $k_5 = (k_3 k_4)^{r/r+1}$.

V. Setting $\bar{k}_0 = k_5 = \bar{k}_1$ one ontains (3.6) as an immediate consequence of (3.7) and (3.8), Step III and (3.24). Note that, since Ψ_2, Ψ_3 are determined by Ψ, which is complementary to Φ all the constants above are determined by Φ and the ordering $\Phi \leqq \Phi_0$, and possibly on a_0 and r. Thus the proof of the lemma is complete.

Proof of Theorem 3.1. The inequalities (3.3) and (3.4) can be deduced from those of the lemma using a standard argument. Thus let $f \in \mathcal{M}^{\Phi}(G)$, and for $A_0 \subset A$, let $\tilde{f}_{A_0}$ be the corresponding function of (3.5) in which $c(i,j,\alpha) = \hat{f}(i,j\,\alpha)$. It is known that such functions as $\{f_{A0}, A_0 \subset A\} \subset \mathcal{M}^{\Phi}$ are dense in the latter. If $\{F(\alpha,\tilde{f}_{A_0}), \alpha \in A_0\}$ is the corresponding function in l^{Ψ}, then a familiar argument shows that $\lim_{A_0 \subset A} J_{\Phi}(\tilde{f}_{A_0} - f) = 0$ and that $\lim_{A_0 \subset A} J_{\Psi}(F_{A_0}) = J_{\Psi}(F)$. From this and the first of the inequalities of (3.6), the inequality (3.3) follows at once. On the other hand, by Fatou's lemma, one has from the second inequality of (3.6)

$$J_{\Psi}(f) \leqq \varliminf_{A_0 \subset A} J_{\Psi}(\tilde{f}_{A_0}) \leqq \bar{k}_1 \lim_{A_0 \subset A} J_{\phi}(F_{A_0}) = \bar{k}_1 J_{\Phi}(F),$$

which is (3.4). This accomplishes the proof of Theorem 3.1.

REMARK. It should be noted that the hypothesis (ii) on $\Psi'(\cdot)$ of Theorem 3.1 (and Lemma 3.2) was used in Step IV of the proof of Lemma 3.2 in obtaining a connection between $J_{\Psi_1}(g)$ and $J_{\Psi}(f_{A_0})$. Note that Young's functions obtained by constructions similar to that of ([5], pp. 28–29) satisfy this hypothesis while Ψ or its complementary function Φ ($\leqq \Phi_0$) need not satisfy the so-called Δ_2-condition. The above proof shows that the result holds under the following slightly more general hypothesis.

Condition (*B*) Let $\Phi \leqq \Phi_0$ be a Young's function such that (i) Φ' is continuous, (ii) if Ψ is its complementary function and L^{Φ}, L^{Ψ} are the Orlicz spaces, on (G,μ), and f_{A_0} is an elementary function (as in (3.5)) on the unit sphere of L^{Ψ}, then there exists a $\Psi_2 \geqq \Psi_1$, where $\Psi_1(x) = \Psi(x^2)$, such that

$$J_{\Psi_1}(\Psi'(f_{A_0})) \leqq kJ_{\Psi_2}(f_{A_0})$$

where k depends only on Ψ and Ψ_2 and the ordering constants, and (iii) there exist $\{\Phi_n\}$ such that $\Phi_n \leqq \Phi$, $\Phi_n \to \Phi_\infty$ in this ordering with $\Phi_\infty(x) \leqq a|x|$, where the ordering constants near zero are bounded below by a positive number.

Note that Condition (B) holds in the case of Theorem 3.1, of which it is an abstraction, and also by certain suitably defined $\Phi(x) = x^p \log_k(x)$, $1 < p < 2$, and $\log_k(x)$ is an iterated logarithm taken k times. This latter is a consequence of some work in [12]. Thus the general form of the result can be stated as follows.

THEOREM 3.3 *Let Φ be a Young's function and G be a compact group with normalized Haar measure μ on it. If $L^{\Phi}(G,\mu)$ is the associated Orlicz space and Condition* (B) *holds for Φ, then for every $f \in \mathcal{M}^{\Phi} \subset L^{\Phi}$, the inequalities* (3.3) *and* (3.4) *hold.*

In fact, with the present hypothesis, (3.22) becomes

$$(3.22)' \qquad 1 \leqq M = M_{\Phi} \leqq \tilde{k}\frac{M_{\Phi_2}}{M_{\Phi}}$$

Thus $1 \leqq M_\Phi \leqq \tilde{k}^{1/2} M_{\Phi_2}^{1/2}$, and using the hypothesis on the Φ_n sequence here one gets (for a fixed $A_0 \subset A$) by iteration,

$$M_\Phi \leqq k_0 \cdot \tilde{k}^{1/2^n} (M_{\Phi n})^{1/2^n}. \tag{3.25}$$

But from (3.11) one notes that M_{Φ_n} is bounded above by $d_0 = \sup\{d(\alpha_0), \alpha_0 \in A_0\} < \infty$ since $\Psi_n^{-1}(\alpha) \to 1$ for $0 < \alpha \leqq 1$ by normalizations and $\Psi_n(1) \to 1$. Hence the above inequality (3.25) shows that as $n \to \infty$, $M_\Phi \leqq k < \infty$. Here $k_0, \tilde{k}$ are constants which depend on Φ and Ψ_2 only. This is (3.24), and hence the rest of the argument holds verbatum.

As promised in the proof of Lemma 3.2, a short proof of the Hausdorff-Young Theorem, slightly different from [4], will be sketched before considering the locally compact case.

Proposition 3.4. *Let* $f \in L^p(G,\mu)$, $1 \leqq p \leqq 2$, *and* $\{F(\alpha,f), \alpha \in A\}$ *be as in* (3.1). *Then* $(p^{-1} + q^{-1} = 1)$

$$\left[\sum_{\alpha \in A} F(\alpha,f)^q d^2(\alpha)\right]^{1/q} \leqq \|f\|_p, \quad \|f\|_q \leqq \left[\sum_{\alpha \in A} F(\alpha,f)^p d^2(\alpha)\right]^{1/p}, \tag{3.26}$$

where the norms on f *are the usual Lebesgue norms.*

Proof. Let $l^q(d^2)$ be the sequence space of qth power summable sequences on A, relative to the weights $\{d^2(\alpha), \alpha \in A\}$ where $d^2(\alpha) \geqq 1$ for all $\alpha \in A$. Let $T: L^p(G,\mu) \to l^q(d^2)$ be an operation defined as $Tf = F(\cdot, f)$. Then T is sublinear, i.e. $|T(af)| = |a|\,|Tf|$, and $|T(f_1 + f_2)| \leqq |(Tf_1)| + |(Tf_2)|$. Moreover, by the Parseval formula,

$$\|Tf\|_2^2 = \sum_{\alpha \in A} F(\alpha,f)^2 d^2(\alpha) = \|f\|_2^2, \tag{3.27}$$

and (by definition),

$$\|Tf\|_q = \begin{cases} \left[\sum_{\alpha \in A} F(\alpha,f)^q d^2(\alpha)\right]^{1/q}, & 1 \leqq q < \infty, \\ \sup_{\alpha \in A} \dfrac{F(\alpha,f)}{d(\alpha)}, & q = \infty. \end{cases}$$

However, the computation given in (3.10) implies

$$\|Tf\|_\infty \leqq \|f\|_1. \tag{3.28}$$

Consequently the Riesz-Thorin theorem for sublinear operators, as extended in [1], implies, from (3.27) and (3.28), $\|Tf\|_q \leqq \|f\|_p$ for $1 \leqq p \leqq 2$. This is the first of (3.26). The second inequality is obtained by an analogous argument (or can be deduced from the first one, cf. [15], Vol. 2, p. 103 or [4]), completing the proof.

4. Locally compact abelian groups. The work of the preceding section enables an extension of Theorem 3.1 to locally compact abelian groups. The techniques of this section seem inadequate for more general groups. However, the present case will be sufficient to an extension of some results of [3] whose work was also one of the original reasons for the present paper.

The main result of this section then can be given as:

THEOREM 4.1. *Let G be a locally compact abelian group and $L^{\Phi}(G,\mu)$ be the Orlicz space on (G,μ). If Φ satisfies the conditions of either Theorem* 3.1 *or of Theorem* 3.3, *and, $f \in \mathcal{M}^{\Phi} \subset L^{\Phi}$, $Tf = \hat{f}$ is the Fourier transform of f, then*

$$J_{\Psi}(Tf) \leqq k_0 J_{\Phi}(f), \tag{4.1}$$

where k_0 is a constant depending only on Φ and the ordering constants in $\Phi \leqq \Phi_0$ (or those connected with the hypothesis of Theorem 3.3). [*As usual, Tf is first defined for $f \in M^{\Phi}(G)$, of compact support and* (4.1) *is established, and then the general case follows by continuity.*]

Proof. Since G is locally compact and abelian, by the structure theorem, G is of the form $G_1 \times R^p$ where R^p is the Euclidean p-space and G_1 is a group which contains a compact subgroup H such that G_1/H is discrete, (cf., [13], p. 40). The proof will be established as follows. The work of Section 3 shows that (4.1) is valid on H. Below it will be shown that the result holds if the group is either discrete or of the form R^p. Since G is (isomorphic to) the direct product of these three groups, the general case then can be deduced. Details will be given in steps.

I. If $G = R^p$, then (4.1) holds. To simplify notation the proof of this step will be given for $p = 1$, where R^1 will be written as R. The case $p > 1$ involves no new ideas or difficulties. The method is an extension of that given in [14]. Thus let $f \in \mathcal{M}^{\Phi}(R)$, and let f have compact support. If $\alpha > 1$ and $\lambda \geqq 1$ are numbers let $n = [\alpha x] - 1$ where $[a]$ denotes the integral part of the real number a. If $a_j = \int_{j/\lambda}^{j+1/\lambda} f(x)\,dx$, set $g_n(x) = \sum_{j=-n}^{n} a_j e^{ijx}$. Then considering $G_\lambda = (-\pi\lambda, \pi\lambda]$ and $d\mu(x) = dx/2\pi\lambda$ as its (normalized) Haar measure, one can apply the results of Section 3 to $g_n(\cdot)$ to get [cf. (3.6) or (3.4)]

$$J_{\Psi}(g_n) \leqq k_1 J_{\Phi}(F_n) \tag{4.2}$$

where, in this case, $F_n = \{a_j, -n \leqq j \leqq n\}$ and k_1 depends only on Φ. On the other hand by (two applications of) Jensen's inequality

$$\lambda\Phi(a_j) \leqq \Phi(\lambda a_j) = \Phi\left(\lambda \int_{j/\lambda}^{j+1/\lambda} f(x)\,dx\right) \leqq \lambda \int_{j/\lambda}^{j+1/\lambda} \Phi(f)\,dx,$$

so that, on adding the first and last terms, one has

$$\sum_{j=-n}^{n} \Phi(a_j) \leqq \int_{-\alpha}^{\alpha} \Phi(f(x))\,dx. \tag{4.3}$$

Replacing f by f/k, $k > 0$, in (4.3) and remembering the definition of norm in Orlicz spaces (see (2.2)) it follows that

$$J_\Phi(F_n) \leqq J_\Phi(f\chi_A) \tag{4.4}$$

where χ_A is the indicator function of $A = (-\alpha,\alpha)$. Hence (4.2) and (4.4) yield for any $\lambda \geqq 1$,

$$J_\Psi(g_n) \leqq k_1 J_\Phi(f\chi_A). \tag{4.5}$$

But as $n \to \infty$ (i.e. $\lambda \to \infty$) $g_n(x) \to \int_A e^{itx} f(t)\,dt$ uniformly in any finite interval of R, (cf. e.g. [14], p. 283) and using the Fatou property of the norm $J_\Psi(\cdot)$ (and the continuity of Ψ), one easily obtains

$$J_\Psi\left(\int_A e^{itx} f(t)\,dt\right) \leqq k_1 J_\Psi(f\chi_A). \tag{4.6}$$

If $F(x,a) = \dfrac{1}{2\pi}\displaystyle\int_{-a}^{a} e^{itx} f(t)dt$, then (4.6) implies for any $b > a > 0$, the following inequality:

$$J_\Psi(F(\cdot\,,b) - F(\cdot\,,a)) \leqq k_1[J_\Phi(f\chi_{(-b,-a)}) + J_\Phi(f\chi_{(a,b)})]. \tag{4.7}$$

Since $f \in \mathscr{M}^\Phi(R)$, one can take the limit as $b \to \infty$, and then $a \to \infty$, inside the norm (cf., e.g., [5], p. 87 or [8], p. 55 about the fact that in $\mathscr{M}^\Phi(R)$ the norm is absolutely continuous—which is used here), so that $F(\cdot\,,a)$ converges in norm $J_\Psi(\cdot)$ to F and

$$J_\Psi(F) \leqq k_1 J_\Phi(f) \tag{4.8}$$

obtains. Clearly $F = \hat{f} = Tf$ here. Thus (4.1) holds for a dense set of functions f (i.e., those with compact supports) in $\mathscr{M}^\Phi(R)$ and hence T has a unique norm preserving extension to all of $\mathscr{M}^\Phi(R)$. Hence (4.1) holds for all f in $\mathscr{M}^\Phi(R)$, as desired.

II. If $G = Z^p$, the discrete group, then (4.1) holds. As before the consideration of the case $p = 1$ (Z^1 will be written as Z) suffices. Then $\hat{G}$, the dual group, is compact, and with the appropriate normalizations of the corresponding Haar measures consider $f \in \mathscr{M}^\Phi(Z)$, the latter space is the closed subspace of l^Φ determined by functions (i.e., sequences) which have only finitely many non-zero values. The proof again is similar to that of Step I above, but is simpler. Thus $\hat{f}(x) = \sum_{n=-\infty}^{\infty} f(n)e^{inx}$, the convergence being uniform for $x \in \hat{G}$, and $\sum_{n=-\infty}^{\infty} \Phi(f(n)) < \infty$. Let $g_n(x) = \sum_{j=-n}^{n} f(j)e^{ijx}$ be defined on $\hat{G}$. Then (3.4) implies

(4.9) $$J_\Psi(g_n) \leqq k_2 J_\phi(F_n) \ (\leqq k_2 J_\phi(f)),$$

where $F_n = \{f(j), -n \leqq j \leqq n\}$, and k_2 depends only on Φ. Since $g_n(x) \to \hat{f}(x)$ as $n \to \infty$, uniformly, one has from (4.9),

(4.10) $$J_\Psi(\hat{f}) \leqq \varliminf_{n\to\infty} J_\Psi(g_n) \leqq k_2 J_\Phi(f),$$

since F_n is also the restriction of f to a finite set of points in Z. Thus (4.1) holds in this case also.

III. The general case is deduced as follows. It has been proved now that, if G is either compact (Theorems 3.1 and 3.3), or (ii) R^n or (iii) Z^p then (4.1) is true. But by the structure theory, a locally compact abelian G is topologically isomorphic with the product of the above three groups. If $f \in \mathscr{M}^\Phi(G)$ and f has compact support, and if Λ_1, Λ_2 and Λ_3 are the character groups of R^n, Z^p and H, then under appropriate identifications (cf. [13], p. 55) these are subgroups of $\hat{G}$ and their union is $\hat{G}$. Moreover, (cf. [13], Theorem 2.7.4) the map $\mathscr{M}^\Phi(G) \to \mathscr{M}^\Phi(G/L)$, where L is a closed subgroup of G (any one of the three above) defined by $f \to F$ for $f \in \mathscr{M}^\Phi(G)$, and $F \in \mathscr{M}^\Phi(G/L)$ such that

$$F(\xi) = \int_L f(xy)\, d\mu_L(y)$$

where $x \in G$ and ξ is the coset of L containing x, (and μ_L is the normalized Haar measure on L) one has $\hat{F}(\hat{x}) = \hat{f}(\hat{x})$ for $\hat{x} \in \Lambda$, the character group of L. It follows from this, that $\hat{F}_i = \hat{f}_{\Lambda_i}$, $i = 1,2,3$ ($\hat{f}_\Lambda$ is $\hat{f}$ restricted to Λ) and that

(4.11) $$J_\Psi(\hat{F}_i) \leqq k_i J_\Phi(f), \quad i = 1,2,3.$$

If $k_0 = 3\max(k_1,k_2,k_3)$ where k_i are the constants that depend only on Φ and the ordering constants, it follows from the definition of the norm $J_\Psi(\cdot)$ (see (2.2)) that (4.11) implies (4.1) with the above k_0. Thus the proof of the theorem is complete.

Some further inequalities obtainable from, and complementary to, those of Theorem 4.1 and of Section 3 together with some related remarks will now be given in the next (and final) section.

5. Related inequalities and remarks. Using the results of the last two sections a new set of inequalities of the Hausdorff-Young type can be obtained by interpolation as shown in the next result. It will then be illuminating to compare these with other (and classical) work.

Let (Φ_i, Ψ_i), $i = 1,2$ be two pairs of Young's complementary functions and $0 \leqq s \leqq 1$. Let Φ_s and Ψ_s be convex functions which are respectively inverse to

(5.1) $$\Phi_s^{-1} = (\Phi_1^{-1})^{1-s}(\Phi_2^{-1})^s, \ \Psi_s^{-1} = (\Psi_1^{-1})^{1-s}(\Psi_2^{-1})^s.$$

That (Φ_s, Ψ_s) is a pair of convex functions satisfying the Young's inequality (2.1), though not necessarily complementary in the sense of Young, was pointed out in ([9], Lemmas 2 and 4).

THEOREM 5.1. *Let Φ be a continuous Young's function with $\Phi(x) > 0$ for $x > 0$. If G is a locally compact abelian group and T denotes the operation of Fourier transform on $\mathcal{M}^{\Phi}(G)$, then T is a densely defined one-to-one closed linear operator on $\mathcal{M}^{\Phi}(G)$ into $\mathcal{M}^{\Psi}(\hat{G})$ where Ψ is the complmentary Young's function to Φ. If for a pair $\Phi = \Phi_i$, $i = 1,2$ the domain of T is all of $\mathcal{M}^{\Phi_i}(G)$, $i = 1,2$, then $T: \mathcal{M}^{\Phi_s}(G) \to M^{\Psi_s}(\hat{G})$, $0 \leqq s \leqq 1$ is a bounded operator and moreover*

$$J_{\Psi_s}(Tf) \leqq k_{s,\Phi_s}(f), \quad f \in \mathcal{M}^{\Phi_s}, \tag{5.2}$$

where (Φ_s, Ψ_s) are defined by (5.1) *and where k_{s,Φ_s} is a constant depending only on s and Φ_1 and Φ_2, the norms $J_{\Phi}(\cdot)$ being those defined in* (2.2). *[As usual $\hat{G}$ always stands for the dual group of G.]*

REMARKS 1. The result of Theorem 4.1 shows that there exist $\Phi_i \leqq \Phi_0$, in plentitude, which satisfy the hypothesis of the theorem preceding the inequality (5.2).

2. The first part of this result and the classical Banach's Open Mapping Theorem, imply that the range of T is either the whole of $\mathcal{M}^{\Psi}(\hat{G})$ or it is of the first category, the latter is a dense subspace if Ψ is also continuous with $\Psi(x) > 0$ for $x > 0$. This result is well-known for the Lebesgue case (cf. [3a]). Using the same technique of [3a], it can be shown that if $\Phi \leqq \Phi_0$, only the second alternative occurs here also.

Proof. Let $P \subset L^{\infty}(G)$ be the set of positive definite functions and let $f \in P \cap \mathcal{M}^{\Phi}(G)$, of compact support. It then is a consequence of the inversion theorem (see, e.g. [7], p. 143) that $\hat{f}$ is also a bounded integrable function on G, and that T is one-to-one. Actually $\hat{f} \in \mathcal{M}^{\Psi}(\hat{G})$. This is immediate if Ψ is continuous. If Ψ is discontinuous and is of the form $\Psi(t) = 0$ for $0 < |t| \leqq t_0 < \infty$ and $= +\infty$ for $|t| > t_0 > 0$, then it is seen that $\mathcal{M}^{\Psi}(\hat{G}) = L^{\Psi}(\hat{G}) = L^{\infty}(\hat{G})$ and the result holds again. Finally if $\Psi(x) > 0$ for $0 < |x| < t_1$, and $\Psi(x) = \infty$ for $|x| > t_1$ so that $L^{\Psi}(\hat{G}) \subset L^{\infty}(\hat{G})$, then define $\bar{\Psi}(x) = \Psi(x)/|x|$, so that $\bar{\Psi}$ is a (not necessarily convex) bounded continuous function on $0 < |x| \leqq t_1 - \delta$ $(\delta \geqq 0)$, and $\bar{\Psi}(x) = \infty$ for $|x| > t_1 + \delta$. Then for some $\alpha > 0$,

$$\int_{\hat{G}} \bar{\Psi}(\alpha\hat{f})\, d\hat{\mu} \leqq \bar{\Psi}(t_1 - \delta) \int_{\hat{G}} |\hat{f}|\, d\hat{\mu} < \infty,$$

where α is chosen such that $\alpha|\hat{f}|(\hat{x}) \leqq t_1 - \delta$, a.e., and $\hat{\mu}$ is the Haar measure on $\hat{G}$. Thus if $\mathcal{D}$ is the linear span of all such f, then $\mathcal{D} \subset \mathcal{M}^{\Phi}(G)$ and $T(\mathcal{D}) \subset \mathcal{M}^{\Psi}(\hat{G})$. Since step functions are dense in $\mathcal{M}^{\Phi}(G)$ and since Φ is continuous, an argument entirely analogous to that in ([7], p. 142) shows that every step function can be approximated by elements of $P \cap \mathcal{M}_{\Phi}(G)$, it follows that $\mathcal{D}$ is also dense in $\mathcal{M}^{\Phi}(G)$.

To show that T is closed, let $\{f_n\} \subset \mathcal{D}$ and $g_n = Tf_n$ be such that $J_{\Phi}(f_n - f) \to 0$, $J_{\Psi}(g_n - g) \to 0$, and let h be in $\mathcal{M}^{\Phi}(\hat{G}) \cap \hat{P}$, with compact support, using an obvious notation. For the proof here the case that $\mathcal{M}^{\Psi} = L^{\infty}$ may be excluded since then the result is known to be true ($L^{\Phi} = \mathcal{M}^{\Phi} = L^1$ holds in that case and the norms are equivalent, so T is actually bounded). So in the remaining cases an argument of the preceding section shows that such h are dense in $\mathcal{M}^{\Phi}(\hat{G})$. Hence

$$\begin{aligned}\int_{\hat{G}} g(\hat{x})h(\hat{x})\,d\hat{\mu}(\hat{x}) &= \lim_{n\to\infty}\int_G\int_{\hat{G}} (x,\hat{x})^* f_n(x)h(\hat{x})\,d\hat{\mu}(\hat{x})\,d\mu(x) \\ &= \lim_{n\to\infty}\int_G \hat{h}(x)f_n(x)\,d\mu(x) = \int_G \hat{h}(x)f(x)\,d\mu(x) \\ &= \int_G\int_{\hat{G}} (x,\hat{x})^* h(\hat{x})f(x)\,d\hat{\mu}(\hat{x})\,d\mu(x) \\ &= \int_{\hat{G}} h(\hat{x})\hat{f}(\hat{x})\,d\hat{\mu}(\hat{x}).\end{aligned}$$

The unexplained notation is self-evident and follows ([7] or [13]). Here the interchange of the limit and the order of integral are justified by the facts that the hypothesis on $g_n h$ and f_n implies that the integrable sequence $\{f_n\hat{h}\}$ is actually Cauchy in $L^1(G,\mu)$, $\hat{h} \in \mathcal{M}^{\Psi}(G)$, and Fubini and Lebesgue-Vitali theorems are applicable. Thus $\int_{\hat{G}}(g(\hat{x}) - (Tf)(\hat{x}))h(\hat{x})\,d\hat{\mu}(\hat{x}) = 0$. Since such h are dense in $\mathcal{M}^{\Phi}(\hat{G})$, it follows that $g - Tf = 0$, a.e., $f \in \mathcal{D}$, and T is closed.

Finally, if the domain of T is $\mathcal{M}^{\Phi_i}(G)$, then by the Closed Graph Theorem, T is bounded: $T: \mathcal{M}^{\Phi_i}(G) \to \mathcal{M}^{\Psi_i}(\hat{G})$, $i = 1,2$ is bounded. Thus $J_{\Psi_i}(Tf) \leqq k_i J_{\Phi}(f)$, $f \in \mathcal{M}^{\Phi_i}(G)$, $i = 1,2$. Now if (Φ_s, Ψ_s) is the pair defined by (5.1), then the inequality (5.2) is a consequence of ([9], Theorem 2) with $k_{s,\Phi_s} = k_1^{1-s}k_2^s$. This completes the proof of the theorem.

In the above result the fact that G is abelian was used crucially even in the very definition of the Fourier transform. However, in Section 3, G is an arbitrary but compact group, and the Fourier transform is defined for every $f \in L^{\Phi}(G) \subset L^1(G)$ It is then natural to examine the corresponding inequality in this case. The first part of the above result need not be considered here. The inequality may be stated as follows:

PROPOSITION 5.2. *Let G be any compact group and (Φ_i, Ψ_i), $i = 1,2$ be Young's complementary functions such that $T: \mathcal{M}^{\Phi_i}(G) \to \mathcal{M}^{\Psi_i}(\hat{G})$, $i = 1,2$ is defined everywhere, where $Tf = F(\cdot, f)$, is the sublinear operator defined by* (3.1). *Then*

(5.3) $$J_{\Psi_s}(F(\cdot,f)) \leqq k_{s,\Phi_s}(f),\ f\in\mathscr{M}^{\Phi_s}(G),\quad 0\leqq s\leqq 1,$$

where k_{s,Φ_s} is a constant which depends only on s and Φ_s, the pair Φ_s, Ψ_s being given by (5.1).

The proof is similar to that of Proposition 3.4. Briefly, the hypothesis implies $J_{\Psi_i}(F(\cdot,f)) \leqq k_i J_{\Phi_i}(f)$, for $f\in\mathscr{M}^{\Phi_i}(G)$, $i = 1,2$. Then to obtain (5.3) one needs an extension of ([9], Theorem 2) for sublinear operators. Even though this is not entirely simple, the corresponding result does hold and this extension yields (5.3). [The needed extension of the interpolation theorem has been worked out recently by Mr. W. T. Kraynek in connection with his thesis at Carnegie.]

SOME REMARKS. (a) The inequalities (5.2) and (5.3) cannot be obtained from the classical Hausdorff-Young inequality (3.26). This is because the Riesz-Thorin convexity theorem gives only the Lebesgue spaces L^p, $1\leqq p\leqq 2$, and its extension ([9], Theorem 2) can be used to give new inequalities (5.2) and (5.3) *only if* an inequality of the Hausdorff-Young type (namely (4.1) or (3.3), (3.4)) for L^Φ-spaces is available. Thus it gives *new* inequalities, as here, whenever the L^Φ so considered is *not* an L^p-space. It should also be remarked that (5.2) and (5.3) yield new inequalities even if only (Φ_2,Ψ_2) is a Young's pair and (Φ_1,Ψ_1) gives a Lebesgue pair. The latter, for instance, is the case if $\Psi_1(x) = 0$ for $0\leqq|x|\leqq t_1$, and $=\infty$ for $|x| > t_1 > 0$. Since then $\mathscr{M}^{\Psi_1}(\hat{G}) = L^{\Psi_1}(\hat{G}) = L^\infty(\hat{G})$ and moreover it is seen that $\|f\|_\infty \leqq t_1 J_{\Psi_1}(f)$, $f\in L^\infty(\hat{G})$. This implies (by the closed graph theorem) that the norms in L^{Ψ_1} and L^∞ are equivalent and that $L^{\Phi_1}(G) = \mathscr{M}^{\Phi_1}(G) = L^1(G)$, and the norms here are also equivalent. Since in this case $T: L^{\Phi_1}(G)\to L^{\Psi_1}(\hat{G})$ is a bounded operator the hypothesis of the theorem for (Φ_1,Ψ_1) is automatically satisfied. *It should be emphasized that, for Theorems* 3.1 *and* 4.1, *it is the method of the classical Hausdorff-Hardy-Littlewood proofs*, which does not use Plancherel's Theorem, *that is seen to be important in this extension.*

(b) An extension of Theorem 4.1 to locally compact unimodular groups can clearly be considered. However, the present methods do not seem to work. It appears that considerable preliminary work (generalizing several results of [6]) is needed even for a precise statement here, and this has yet to be done. The need for inequalities of the type (5.2), and (5.3), was noted in applications in the past (cf., e.g., [10], Theorem 5). Moreover, most of the work presented here is useful in extending the work of [3].

(c) In an unpublished Ph.D. thesis, written under Professor A. Zygmund at the University of Chicago, W. J. Riordan has proved in 1957, a result similar to that of Theorem 4.1 (with $G = R$ there) if $\Phi(x) = |x|^p L(x)$, $1 < p < 2$, where $L(\cdot)$ is a product of the (a finite number of iterates of) logarithmic functions such that $\Phi(\cdot)$ is a Young's function (cf. [12]), then $\Psi(x) = |x|^q L_1(x)$ where $p^{-1}+q^{-1} = 1$ and $L_1(x) = \left[L\left(\frac{1}{x}\right)\right]^{q/p}$ for large x is defined such that $\Psi(\cdot)$

is a Young's function. (Φ, Ψ) need not be complementary here [as is the case with (Φ_s, Ψ_s) of (5.1)], unlike that of Theorem 4.1. It seems likely, however, that the complementary function of Φ and the above Ψ define the same Orlicz space with equivalent norms (as is the case with (Φ_s, Ψ_s), cf. [9], Lemma 4), though this is not immediate from Riordan's work. His proof is based on an interesting extension of the Marcinkiewicz's interpolation theorem for certain Orlicz spaces which was first established in the thesis and from which an inequality of the type (4.1) was deduced. Thus Riordan's result and Theorem 4.1 may complement each other in some cases and may coincide in others whenever they are comparable.

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